1.0 Introduction to methods of structural analysis

It is well known that there are two basic methods of structural analysis of indeterminate structures which are:

1. Flexibility methods (also known as force methods, compatibility methods or the methods of consistent deformation) and
2. Displacement methods (also known as stiffness or equilibrium method)

Each method involves the combination of a particular solution which is obtained by making the structure statically determinate, and a complementary solution in which the effects of each individual modification is assessed. In the force methods, the behaviour of the structure is considered in terms of unknown forces, while in the stiffness method, the behaviour of the structure is considered in terms of unknown displacements. By implication, both analysis methods always involve reducing the structure to a basic system (a determinate system). In the force method, the basic system involves the removal of redundant forces, while the stiffness method involves restraining the joints of the structure against displacement.

Whenever we are using the force method, our basic system is dependent on the degree of static indeterminacy. A basic system is a system that is statically determinate and stable. The choice of redundant constraints to remove is based on geometrical configuration, stability of the structure, and ease of analysis. Degree of static indeterminacy can be computed using:

\[ D = (3M + R) - 3N \]  \[ (1) \]

Where;
- \( D \) = Degree of static indeterminacy
- \( M \) = Number of members
- \( R \) = Number of constraints (reactions)
- \( N \) = Number of nodes

Alternatively, the following relation can be used:

\[ D = R - 3 - S \]  \[ (2) \]

Where;
- \( D \) = Degree of static indeterminacy
- \( R \) = Number of constraints (reactions)
- \( S \) = Number of special conditions (such as an internal hinge)

After selecting a good basic system, we now replace the redundant supports with unit loads and evaluate them one after another. In this step, we calculate the deflection corresponding to each redundant force separately due to applied loading and other redundant forces from force-displacement relations. Deflection due to redundant force cannot be obtained without knowing the magnitude of the redundant force. Therefore, we apply a unit load in the direction of redundant force and determine the
corresponding deflection. Since the principle of superposition is valid in elastic analysis, the
deflections due to redundant force can be obtained by multiplying the unknown redundant with the
deflection obtained from applying unit value of force. Now, we calculate the total deflection due to
the applied loading and the redundant force by applying the principle of superposition which must be
compatible with the existing boundary condition.

For more than one set of redundant forces, we construct a set of simultaneous equations with the
redundant forces as unknowns and flexibility coefficients as coefficients of the equations. These
flexibility coefficients are also called the influence coefficients. The total number of equations equals
the number of unknown redundant forces. These set of simultaneous equations are called the canonical equations, and they are of the form shown below;

\[ \delta_{11}X_1 + \delta_{12}X_2 + ... + \delta_{1n}X_n + \Delta_1P = 0 \]
\[ \delta_{21}X_1 + \delta_{22}X_2 + ... + \delta_{2n}X_n + \Delta_2P = 0 \]
\[ \vdots \quad \vdots \quad \vdots \quad \vdots \]
\[ \Delta_{n1}X_1 + \delta_{n2}X_2 + ... + \delta_{nn}X_n + \Delta_nP = 0 \]  \[ (3) \]

Where:
\( \delta_{ij} \) = deformation at point \( i \) due to a unit load at point \( j \)
\( X_i \) = Redundant force at point \( i \)
\( \Delta_iP \) = Deformation at point \( i \) due to externally applied load

Mohr’s integral \( \delta_{ij} = \int_0^L \frac{M}{EI} \, ds \)  \[ (4) \]

Where:
\( \delta_{ij} \) = deformation at point \( i \) due to a unit load at point \( j \)
\( M \) = Moment due to externally applied load or another redundant load
\( \bar{M} \) = Moment due to unit load (virtual load)

### 1.1 Vereshchagin’s rule

The easiest method of calculating influence coefficients is the graphical method. It is handier than
using the Mohr’s integral and it is based on Vereshchagin’s rule. You must know how to draw
bending moment diagrams very well (with the appropriate sign convention) before this method can
work for you. It is all about the combination of bending moment diagrams of both the externally
applied load, and the redundant loads. By implication, you must elaborately draw the bending moment
diagram of each redundant load, and that of the externally applied load. To show how this works, let
us go through a simple overview of how Vereshchagin’s rule works by considering the two shapes
shown below.

![Moment Diagram](image)

**Figure 1.1: Illustration on diagram combination using Vereshchagin’s rule**
Let us try to combine the two shapes in Figure 1.1 using Vereschagin’s rule, assuming that the original moment diagram (the one up) is obtained from the internal stresses of a simply supported beam that is loaded uniformly, and the linearly diagram due to a unit point load applied at the left hand side of the support.

We all know that the maximum moment of the original diagram will occur at the mid-span and the value is equal to \( \frac{qL^2}{8} = Mc \), where \( q \) is the externally applied uniform load and \( L \) is the length of the span. Also note that the centroid of the shape lies at \( x = \frac{L}{2} \) (parabola).

For the linearly diagram (the one below), the ‘maximum moment’ will occur at the right support with a value that is equal to the span of the section, say \( L = M_A \) (since a unit load is applied). The ordinate of the linear diagram that will correspond to the centroid of the original diagram will be the value of \( y \) in the triangle, when \( x = \frac{L}{2} \).

Therefore, the influence coefficient for the above two shapes can be computed using:

\[
\delta = \frac{1}{3} \times \frac{M_A}{M_c} \times M_c \times L
\]

Now, lists of equations that we are going to use to combine the diagrams that we will generate in our analysis have been presented in Table 1. They are all derived the same way that the one above has been derived, and I advise you to prove all of them by yourself using Vereschagin’s rule. The basic thing is to know the area of your diagrams and their centroid, and then the rest is simple mathematics. If you are having hard time, I advise that you go back to your Statics textbook.

There are no hard rules about diagram combination, anyone can combine the other. I will like us also to understand that the diagrams combining each other must be equal in length, otherwise you are expected to split the diagram from their points of contraflexure, or from other convenient point as the case may be. Also, in moment diagrams such as that produced when a cantilever with a uniformly distributed load is propped at the free end, it is most convenient to split it into two diagrams as shown below, paying great attention to the signs. You will fully understand why such splitting is possible when you try to solve this using the integration method.
Table 1.1: Influence co-efficient formulae for combining some basic shapes

<table>
<thead>
<tr>
<th>Shape</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Shape 1" /></td>
<td>( \frac{1}{3} MA \cdot MC \cdot L' )</td>
</tr>
<tr>
<td><img src="image2" alt="Shape 2" /></td>
<td>( \frac{1}{3} MA \cdot MAL' )</td>
</tr>
<tr>
<td><img src="image3" alt="Shape 3" /></td>
<td>( \frac{1}{6} MA(2MA + MB) \cdot L' )</td>
</tr>
<tr>
<td><img src="image4" alt="Shape 4" /></td>
<td>( \frac{1}{6} MA(2MA + MB - MB) \cdot MA + 2MB \cdot L' )</td>
</tr>
</tbody>
</table>

Example 1.1

The frame shown below is supported with rollers at A, C and D, and pinned at point B. It is loaded as shown below, and all columns have a cross-section of 30cm x 30cm, while the beams have a cross-section of 45cm x 30cm. Draw the bending moment diagram due to the externally applied load. Where necessary take \( E = 21.5 \text{ KN/mm}^2 \).
Solution

Step 1: Determine the degree of static indeterminacy.

\[ D = R - e - s \]  where;
\[ D = \text{Degree of indeterminacy} \]
\[ R = \text{number of support reactions in the system} = 5 \]
\[ E = \text{number of equations of equilibrium} = 3 \]
\[ S = \text{any special condition, e.g. internal hinge} = 0 \]

Therefore, \[ N = 5 - 3 - 0 = 2 \]

Therefore, the structure is indeterminate to the 2nd degree.

And we are expected to remove two redundant supports, and ultimately solve a simultaneous equation containing two unknowns in our canonical equation.

Step 2: Reduce the structure to a basic system

As we realised from the first step, we will need to remove two redundant supports in order to make the structure statically determinate. However, we must ensure that the selected system must be stable. If we remove supports A and D, we will obtain a good basic system. If we remove supports C and D, we will also obtain a good basic system, likewise A and C. If we remove the two reactive forces at support B, we will not obtain a good basic system. If the structure were symmetrical enough in terms of support, it is possible to split the structure from the middle. So watch out for the stability of a structure when selecting a basic system.

So in this example, we are removing the reactive forces at supports C and D and replacing them \( X_1 \) and \( X_2 \) which will be assigned unit values. The basic system is shown below.

![Basic System Diagram](image)

Step 3: Analysis of the various load cases (\( X_1 \), \( X_2 \) and externally applied load)

The values \( X_1 \) and \( X_2 \) represent the redundant forces that have been removed from the system, and they will be assigned unit values in order for us to progress in our analysis. We are going to treat each of them as independent load cases on the structure. Our main interest will be on the moment diagrams that they produce, because we will neglect shear and axial forces.
Case 1 \((X_1 = 1.0, X_2 = 0)\)

Support reactions:

Let \(\sum M_A = 0\): \((1 \times 9) + (By \times 5) = 0\)

Therefore, \(By = -\frac{9}{5}\)

Let \(\sum M_B = 0\): \(-(1 \times 4) + (Ay \times 5) = 0\)

Therefore, \(Ay = \frac{4}{5}\)

Let \(\sum F_X = 0\):

\(Bx = 0\)

Case 2 \((X_1 = 0, X_2 = 1.0)\)

Support reactions:

Let \(\sum M_A = 0\): \((1 \times 14) + (By \times 5) = 0\)

Therefore, \(By = -\frac{14}{5}\)

Let \(\sum M_B = 0\): \(-(1 \times 9) + (Ay \times 5) = 0\)

Therefore, \(Ay = \frac{9}{5}\)

Let \(\sum F_X = 0\):

\(Bx = 0\)

Loading the basic system with externally applied load

Support reaction:

Let \(\sum M_A = 0\): \(- (6 \times 11.5) - (10 \times 7) - \left(\frac{5 \times 5^2}{2}\right) + (5 \times By) = 0\)

Therefore, \(By = 40.3 \text{ kN}\)

Let \(\sum M_B = 0\): \((Ay \times 5) + (10 \times 2) - \left(\frac{5 \times 5^2}{2}\right) + (6 \times 6.5) = 0\)

Therefore, \(Ay = 0.7 \text{ kN}\)

Let \(\sum F_X = 0\):

\(Bx = 0\)
Elaborate analysis of the bending moments due to externally applied load on the basic system

\[ M_A = M_B = M_C = M_D = 0 \text{ (all simply supported)} \]

Section A-1 \((0 \leq y \leq 4.0 \text{ m})\)

Zero moment (since \(A_x = 0\))

Section 1 - 2 \((0 \leq x \leq 5.0 \text{ m})\)

\[ M_X = (Ay \times x) - \left( \frac{5}{2} x^2 \right) = 0.7x - 2.5x^2 \]

At \(x = 0; M_1 = 0\)

At \(x = 5\text{m}; M_2 = 0.7(5) - 2.5(5)^2 = -59.00 \text{ KN.m}\)

Section B-2 \((0 \leq y \leq 4.0 \text{ m})\)

Zero moment (since \(B_x = 0\))

Section 2 - 3 \((4.0 \leq y \leq 6.0\text{ m})\)

Since \(B_x = 0\), the sum of the moments from Span (1-2) will act as a couple on the section (2-3). As a result, \(M_2 = M_1 = -59.00 \text{ KN.m}\)

Section 3 - \(E^R(5 \leq x \leq 7\text{ m})\)

\[ M_X = (Ay \times x) - [(5 \times 5) \times (x - 2.5)] + By(x - 5) \]

\[ M_X = (0.7x) - 25(x - 2.5) + 40.3(x - 5) \]

At \(x = 5\text{m}; M^{R}_{2} = (0.7 \times 5) - 25(5 - 2.5) + 40.3(5 - 5) = -59.00 \text{ KN.m}\)

At \(x = 7\text{m}; M^{L}_{2} = (0.7 \times 7) - 25(7 - 2.5) + 40.3(7 - 5) = -27.00 \text{ KN.m}\)

Section \(E^R - 4(7 \leq x \leq 9\text{ m})\)

\[ M_X = (Ay \times x) - [(5 \times 5) \times (x - 2.5)] + By(x - 5) - 10(x - 7) \]

\[ M_X = (0.7x) - 25(x - 2.5) + 40.3(x - 5) - 10(x - 7) \]

At \(x = 7\text{m}; M^{R}_{2} = (0.7 \times 7) - 25(7 - 2.5) + 40.3(7 - 5) - 10(7 - 7) = -27.00 \text{ KN.m}\)

At \(x = 9\text{m}; M^{L}_{2} = (0.7 \times 9) - 25(9 - 2.5) + 40.3(9 - 5) - 10(9 - 7) = -15.00 \text{ KN.m}\)

Section C - 5 \((0 \leq y \leq 4.0\text{ m})\)

Zero moment (since \(C_x = 0\))

Section 4 - 5 \((4 \leq y \leq 6.0\text{ m})\)

Since section 5-C has no external horizontal load or its equivalent, the sum of the moments from Span (3-4) will act as a couple on the section (4-5). As a result, \(M^{R}_4 = M^{UP}_5 = -15.00 \text{ KN.m}\)

Section 5 - \(F(9 \leq x \leq 11.5\text{ m})\)

\[ M_X = (Ay \times x) - [(5 \times 5)(x - 2.5)] + By(x - 5) - 10(x - 7) \]

\[ M_X = (0.7x) - 25(x - 2.5) + 40.3(x - 5) - 10(x - 7) \]

At \(x = 9\text{m}; M^{R}_{2} = (0.7 \times 9) - 25(9 - 2.5) + 40.3(9 - 5) - 10(9 - 7) = -15.00 \text{ KN.m}\)

At \(x = 11.5\text{m}; M^{L}_{2} = (0.7 \times 11.5) - 25(11.5 - 2.5) + 40.3(11.5 - 5) - 10(11.5 - 7) = 0\)
This concludes the analysis for obtaining the values of the bending moments across different sections of the structure. It could have been quite faster if we had analysed some of the sections coming from the right. You should know that all sections are cut, and all horizontal distances were with respect to support A. In essence, all equations are generated with respect to support A. It is possible however to plot the bending moment diagram by just punching your calculator directly without showing the exact details. This is often permitted when using this method. It is the accuracy the diagram that matters.

The bending moment diagram is shown below.

**Step 4: Computation of influence coefficients**

**Geometrical properties of the sections**

Columns = 30cm x 30cm = 0.3m x 0.3m

Moment of inertia of column $I_C = \frac{0.3 \times 0.3^3}{12} = 6.75 \times 10^{-4} \text{ m}^4$

Beams = 45cm x 30cm = 0.45m x 0.3m

Moment of inertia of column $I_B = \frac{0.3 \times 0.45^3}{12} = 2.278 \times 10^{-3} \text{ m}^4$

In this example, I desire to work with respect to $EI_C$. Since E is constant, we are neglecting it so far. $\frac{I_C}{I_B} = \frac{6.75 \times 10^{-4}}{2.278 \times 10^{-3}} = 0.2963$

Influence coefficients $EI_C \delta = \int_0^L M_B \frac{I_C}{I_B} ds = L' \int M_B ds$ Where; $L' = \frac{I_C}{I_B}$

To take care of this, we multiply beam members with $\frac{I_C}{I_B} = 0.2963$ and column members by 1.000.

$\delta_{11} =$ Deflection at point 1 due to unit load at point 1
\[ \delta_{11} = \left( \frac{1}{3} \times 4 \times 4 \times 5 \times 0.2963 \right) + (4\times 4 \times 2) + \left( \frac{1}{3} \times 4 \times 4 \times 4 \times 0.2963 \right) \]
\[ \delta_{11} = 7.901 + 32 + 6.321 = 46.222 \]

\[ \delta_{12} = \delta_{12} \quad \text{Deflection at point 1 due to unit load at point 2} \]

\[ \delta_{12} = \left( \frac{1}{3} \times 4 \times 9 \times 5 \times 0.2963 \right) + (4\times 9 \times 2) + \frac{1}{6} \times \left[ 4 \times ((2 \times 9) + 5) \times 4 \times 0.2963 \right] \]
\[ \delta_{12} = \delta_{12} = 17.778 + 72 + 18.173 = 107.951 \]

\[ \delta_{22} = \text{Deflection at point 2 due to unit load at point 2} \]

\[ \delta_{22} = \left( \frac{1}{3} \times 9 \times 9 \times 5 \times 0.2963 \right) + (9\times 9 \times 2) + \frac{1}{6} \times \left[ 5 \times (2(5) + 9) + 9 \times (2(9) + 5) \times 4 \times 0.2963 \right] \]
\[ \delta_{22} = 40.000 + 162 + 59.655 + 50 + 12.346 = 324.001 \]

\[ \Delta IP: \text{Deformation at point 1 due to externally applied load} \]

Note that the moment diagram for span 1-2 due to the external loads has been split to obtain the first two diagrams in the combination (see section 1.1).

\[ \Delta IP = \left( \frac{1}{3} \times 15.625 \times 4 \times 5 \times 0.2963 \right) - \left( \frac{1}{3} \times 59 \times 4 \times 5 \times 0.2963 \right) - (59 \times 4 \times 2) - \]
\[ \frac{1}{6} \times [59 \times (2(4) + 2) + 27 \times (2(2) + 4) \times 2 \times 0.2963] - \frac{1}{6} \times [2 \times (2(27) + 15) \times 2 \times 0.2963] \]

N/B: Realise that the moment diagram in member 3 – 4 was split into two respective trapeziums.
Δ₂P: Deformation at point 2 due to externally applied load

\[ \Delta \Delta P = \left( \frac{1}{3} \times 15.625 \times 9 \times 5 \times 0.2963 \right) - \left( \frac{1}{3} \times 59 \times 9 \times 5 \times 0.2963 \right) - (59 \times 9 \times 2) \\
- \frac{1}{6} \times [59(2(9) + 7) + 27(2(7) + 9) \times 2 \times 0.2963] - \frac{1}{6} \times [27(2(9) + 5) + 15(2(5) + 9) \times 2 \times 0.2963] - \\
(15 \times 5 \times 2) - \frac{1}{6} \times [15(2(5) + 2.5) \times 2.5 \times 0.2963] \\
\Delta \Delta P = 69.445 - 262.226 - 1062 - 207.015 - 89.483 - 150 - 23.148 = -1724.427 \\

Step 5: Canonical equations

The appropriate canonical equation is

\[ \delta_{11}X_1 + \delta_{12}X_2 + \Delta \Delta P = 0 \\
\delta_{21}X_1 + \delta_{22}X_2 = 0 \\
46.222X_1 + 107.591X_2 = 650.917 \\
107.591X_1 + 324.001X_2 = 1724.427 \\
\]

Solving simultaneously,

\[ X_1 = 7.46 \text{ KN} \\
X_2 = 2.85 \text{ KN} \\
\]

Note now that the values \( X_1 \) and \( X_2 \) represent the actual vertical support reactions occurring at roller supports C and D. Since we have known these values, the structure has become determinate, and we can analyse it by the law of statics, or progress using the force method. In this example, we are using the force method to obtain the final moment diagram, and not the shear and axial forces (see further examples in other posts for shear and axial).

Step 6: Final internal stresses (moment only in this example)

\[ M_0 = \bar{M}_1X_1 + \bar{M}_2X_2 + M_0 \\
M_A = M_B = M_C = M_D = 0 \\
M_1 = 0 \\
M_2^L = (7.46 \times 4) + (9 \times 2.85) - 59 = -3.51 \text{ KN.m} \]
\[ M_2^B = 0 \]
\[ M_{2\text{UP}} = (7.46 \times 4) + (9 \times 2.85) - 59 = -3.51 \text{ KN.m} \]
\[ M_3^B = M_3^L = (7.46 \times 4) + (9 \times 2.85) - 59 = -3.51 \text{ KN.m} \]
\[ M_{10\text{KN}} = (7.46 \times 2) + (2.85 \times 7) - 27 = +7.87 \text{ KN.m} \]
\[ M_4^L = M_4^B = (7.46 \times 0) + (5 \times 2.85) - 15 = -0.75 \text{ KN.m} \]
\[ M_5^B = M_5^R = (7.46 \times 0) + (5 \times 2.85) - 15 = -0.75 \text{ KN.m} \]
\[ M_{6\text{KN}} = (7.46 \times 0) + (2.85 \times 2.5) - 0 = +7.125 \text{ KN.m} \]
\[ M_6 = 0 \]

**Step 7:** Plot the final internal stresses diagram (moment only in this example)